PRINCIPLES OF ANALYSIS
LECTURE 21 - PROPERTIES OF THE DERIVATIVE

PAUL L. BAILEY

1. Leibnitz Notation

Let $f : D \to \mathbb{R}$ with x_0 an accumulation point of D.

Let $\Delta x = x - x_0$; viewing x_0 as fixed, this is implicitly a function of x. Let $\Delta f = f(x) - f(x_0)$; viewing f as fixed, this is also a function of x.

Now x goes to x_0, we see that Δx goes to 0. Thus

$$\lim_{x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Thus we may define the derivative to be

$$\frac{df}{dx} = \lim_{x \to 0} \frac{\Delta f}{\Delta x}.$$

Moreover, in Leibnitz notation, it is traditional to start with a function whose name is y instead of f, so this becomes

$$\frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}.$$

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2. Chain Rule

If we have two lines $y = m_1 z + b_1$ and $z = m_2 x + b_2$ and compose them, we obtain

$$y = m_1 m_2 x + (m_1 b_2 + b_1),$$

a line with slope $m_1 m_2$. Since we view a differentiable function as a function which is approximately a line whose slope is the derivative, we guess that the derivative of a composition is the product of the derivatives.

Suppose that y is a function of u and u is a function of x. Then we may attempt to right

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Then $\Delta u \to 0$ as $\Delta x \to 0$, so taking the limit of both sides we would arrive at

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The problem with this reasoning is that Δu may be zero even when Δx is nonzero. We have to get around this problem.

Proposition 1 (Chain Rule)**.** *Let $X, Y \subset \mathbb{R}$ with $x_0 \in D$ an accumulation point of X and $y_0 \in Y$ and accumulation point of Y. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ with $f(X) \subset Y$ and $f(x_0) = y_0$. If f is differentiable at x_0 and g is differentiable at y_0, then $g \circ f$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0).$$

Proof. Define a function $U : X \to \mathbb{R}$ by $U(x) = \frac{g(f(x)) - g(f(x_0))}{x - x_0}$; we wish to show that $U(x)$ has a limit at $x = x_0$, and that $\lim_{x \to x_0} U(x) = g'(y_0) f'(x_0)$.

Define $h : Y \to \mathbb{R}$ by

$$h(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} & \text{if } y \neq y_0; \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Since g is differentiable at y_0, we have $\lim_{y \to y_0} h(y) = g'(y_0) = h(y_0)$, so h is continuous at y_0. Since f is differentiable at x_0, it is continuous at x_0, and since $f(x_0) = y_0$, then $h \circ f$ is continuous at x_0.

Set $T(x) = \frac{f(x) - f(x_0)}{x - x_0}$. We claim that for $x \in D \setminus \{x_0\}$, we have $U(x) = h(f(x)) \cdot T(x)$. If $f(x) = f(x_0)$, then $g(f(x)) = g(f(x_0)) = g(y_0)$. In this case, $U(x) = 0$ and $h(f(x)) \cdot T(x) = 0$. Otherwise, $U(x) = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} = h(f(x)) \cdot T(x)$.

Now take the limit to see that

$$\lim_{x \to x_0} U(x) = \lim_{x \to x_0} h(f(x)) \lim_{x \to x_0} T(x) = g'(y_0) f'(x_0).$$

□

3. Extrema

Let $f : D \to \mathbb{R}$ and let $x_0 \in D$.

We call x_0 a *global maximum* [respectively *global minimum*] of f if $f(x) \leq f(x_0)$ [respectively $f(x) \geq f(x_0)$] for all $x \in [a, b]$. If x_0 is a global minimum or a global maximum, it is called a *global extremum*.

Proposition 2. *Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$ be continuous. If D is compact, then there exist $x_1, x_2 \in D$ such that x_1 is a global minimum of f and x_2 is a global maximum of f.*

Proof. Since D is compact and f is continuous, then $f(D)$ is compact. Thus $\inf f(D) \in f(D)$ and $\sup f(D) \in f(D)$. So there exist $x_1, x_2 \in D$ such that $f(x_1) = \inf f(D)$ and $f(x_2) = \sup f(D)$. Then x_1 is a global minimum and x_2 is a global maximum. □

We call x_0 a *local maximum* [respectively *local minimum*] of f if there exists a neighborhood Q of x_0 such that for $x \in Q \cap D$ we have $f(x) \leq f(x_0)$ [respectively $f(x) \geq f(x_0)$]. If x_0 is a local minimum or a local maximum, it is called a *local extremum*.

Proposition 3. *Let $f : [a, b] \to \mathbb{R}$ and let $x_0 \in [a, b]$ be a local extremum of f. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose that x_0 is a local maximum. Then there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all x satisfying $|x - x_0| < \delta$.

Set $T(x) = \frac{f(x) - f(x_0)}{x - x_0}$ for $x \in D \setminus \{x_0\}$. Since f is differentiable at x_0, $\lim_{n \to \infty} T(x_n) = f'(x_0)$ for every sequence from $(x - \delta, x + \delta)$ which converges to x_0.

Note that the numerator of $T(x)$ is negative for x near x_0. For $x_n = x - \frac{\delta}{n}$, we see that $T(x_n) \geq 0$, so $f'(x_0) \geq 0$. However, for $x_n = x + \frac{\delta}{n}$, we have $T(x_n) \leq 0$, so $f'(x_0) \leq 0$. This shows that $f'(x_0) = 0$. □

4. Rolle's Theorem

Proposition 4 (Rolle's Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. Then if* $f(a) = f(b)$*, there exists* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

Proof. Since $[a, b]$ is compact, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1)$ is a global minimum and $f(x_2)$ is a global maximum. If $f(x_1) = f(x_2)$, then f is constant, and $f'(x) = 0$ for every $x \in [a, b]$. Otherwise, either $x_1 \neq f(a)$ or $x_2 \neq f(a)$. Therefore either $x_1 \in (a, b)$ or $x_2 \in (a, b)$. If $x_1 \in (a, b)$, then x_1 is a local minimum, and $f'(x_1) = 0$. If $x_2 \in (a, b)$, then x_2 is a local maximum, and $f'(x_2) = 0$. □

Proposition 5 (Mean Value Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. Then there exists* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(b)}{b-a}$*.*

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - \frac{x-a}{b-a}(f(b) - f(a))$. There g is continuous on $[a, b]$ and differentiable on (a, b), and we compute that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

However, $g(a) = f(a)$ and $g(b) = f(a)$. By Rolle's Theorem, there exists $c \in [a, b]$ such that $g'(c) = 0$, so $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

5. Inverse Function Theorem

Proposition 6. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) \neq 0$ for $x \in [a,b]$, then f is injective, and the inverse of f is differentiable on $f((a,b))$, with*

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

for every $x \in (a,b)$.

Proof. Suppose that f is not injective. Then there exist $x_1, x_2 \in [a,b]$ such that $f(x_1) = f(x_2)$. By Rolle's Theorem, there exists $c \in [x_1, x_2]$ such that $f'(c) = 0$; this violates the hypothesis. Thus f is injective.

We have seen that a continuous bijective function on a compact set has a continuous inverse; since $[a,b]$ is compact, $f^{-1} : f([a,b]) \to [a,b]$ is continuous.

Now let $y_0 \in f((a,b))$, and let $\{y_n\}_{n=1}^{\infty}$ be an arbitary sequence from $f((a,b)) \smallsetminus \{y_0\}$ which converges to y_0. Set $x_0 = f^{-1}(y_0)$ and $x_n = f^{-1}(y_n)$. It suffices to show that $\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$.

Since f^{-1} is continuous, we see that $\lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0)$, that is, $\lim_{n\to\infty} x_n = x_0$. Thus, since f is differentiable at x_0, we have

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0).$$

Since f is injective, $f(x_n) - f(x_0) \neq 0$ unless $x_n = x_0$, so by a property of limits of sequences we have

$$\frac{1}{f'(x_0)} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}.$$

□

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu